Calcolare l'insieme delle primitive della seguente funzione

$$\frac{\operatorname{sen}(2x)}{\cos x \cos 2x - \cos^3 x}$$

Risulta:

$$\frac{\operatorname{sen} 2x}{\cos x \cos(2x) - \cos^3 x} = \frac{2 \operatorname{sen} x \cos x}{\cos x(\cos^2 x - \operatorname{sen}^2 x) - \cos^3 x} = \frac{2 \operatorname{sen} x \cos x}{\cancel{\cos^3 x} - \cos x \operatorname{sen}^2 x - \cancel{\cos^3 x}}$$

$$= \frac{2 \cancel{\cos x} \cancel{\operatorname{sen} x}}{- \cancel{\cos x} \operatorname{sen}^{\cancel{2}} x} = \frac{-2}{\operatorname{sen} x} \qquad \forall x \neq 0 + k\pi$$

Per le formule di razionalizzazione:

$$\operatorname{sen} x = \frac{2 \operatorname{tg} \frac{x}{2}}{1 + \operatorname{tg}^2 \frac{x}{2}} \qquad \forall x \neq \pi + 2k\pi$$

. Posto $t = \operatorname{tg} \frac{x}{2}$

$$dt = d\left(\operatorname{tg} \frac{x}{2}\right) = \frac{1}{2}\left(1 + \operatorname{tg}^2 \frac{x}{2}\right) dx$$

$$\int \frac{-2}{\operatorname{sen} x}\,dx = -2\int \frac{1}{\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}\cdot\frac{\frac{1}{2}\left(1+\operatorname{tg}^2\frac{x}{2}\right)}{\frac{1}{2}\left(1+\operatorname{tg}^2\frac{x}{2}\right)}\,dx = -2\int\frac{dt}{t}$$

dove $t=\operatorname{tg}\frac{x}{2}$

$$= \Big\{-2\log|t|+c\Big\}_{t=\operatorname{tg}\frac{x}{2}} = \Big\{-2\log\left|\operatorname{tg}\frac{x}{2}\right|+c\Big\}$$

Si poteva procedere anche nel seguente modo

$$\int \frac{\operatorname{sen}(2x)}{\cos x\cdot\cos(2x)-\cos^3 x}\,dx = \int\frac{2\operatorname{sen}x\,\cancel{\cos x}}{\cancel{\cos x}(2\cos^2 x-1)-\cos^{\cancel{3}\,2} x} = \int\frac{2\operatorname{sen}x}{\cos^2 x-1}$$

Posto $t=\cos x$ $\qquad dt = d(\cos x) = -\operatorname{sen}x\,dx$

$$\int \frac{2\,\operatorname{sen} x}{\cos^2 x - 1}\,dx = -\int \frac{-2\,\operatorname{sen} x\,dx}{\cos^2 x - 1} = \int \frac{2\,dt}{t^2 - 1}$$

$$\frac{2}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} \rightarrow \frac{At + A + Bt - B}{(t-1)(t+1)} \rightarrow \begin{cases} A+B=0 \\ A-B=2 \end{cases}$$

$$\Rightarrow \begin{cases} A=-B \longrightarrow A=1 \\ -2B=2 \rightarrow B=-1 \end{cases}$$

$$\int \frac{2}{t^2-1}\,dt = \int \frac{1}{t-1}\,dt + \int \frac{-1}{t+1}\,dt = \left\{ \log|t-1| - \log|t+1| + c \right\}$$

$t = \cos x$

$$= \left\{ \log\left|\frac{\cos x - 1}{\cos x + 1}\right| + c \right\}$$

Osserva che

$$D\left[\log\left|\frac{\cos x-1}{\cos x+1}\right|\right] = \underbrace{\frac{-2\,\mathrm{sen}\,x}{\cos^2 x-1} = \frac{-2}{\mathrm{sen}\,x}}_{\substack{\| \\ \dfrac{\mathrm{sen}\,2x}{\cos x\cos(2x)-\cos^3 x}}} = D\left[-2\log\left|\mathrm{tg}\,\frac{x}{2}\right|\right]$$

Quindi

entrambe le funzioni: $\log\left|\frac{\cos x-1}{\cos x+1}\right|$ e $-2\log\left|\mathrm{tg}\,\frac{x}{2}\right|$

sono primitive della funzione $\frac{\mathrm{sen}\,2x}{\cos x\cos(2x)-\cos^3 x}$.